International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)
DOI: 10.1007/s10773-005-8015-1

Open Problems for Sequential Effect Algebras

Stan Gudder[1]

A sequential effect algebra (SEA) is an effect algebra on which a sequential product with certain natural properties is defined. In such structures, we can study combinations of simple measurements that are series as well as parallel. This article presents some open problems for SEAs together with background material, comments and partial results. Two examples of open problems are the following: is $A \circ B = A^{1/2}BA^{1/2}$ the only sequential product on a Hilbert space SEA? It is known that the sharp elements of a SEA form an orthomodular poset. Is every orthomodular poset isomorphic to the set of sharp elements for some SEA?

KEY WORDS: effect algebras; sequential effect algebras; positive operators; quantum effects.

PACS: 02.10.+w, 02.30.+g, 03.65.Bz.

1. INTRODUCTION

It has been said that a mathematical theory is mature if it contains at least 50 important theorems. Using this criterion, we can surely say that each of the main mathematical disciplines, algebra, geometry, topology and analysis are mature. Even the subdisciplines, algebraic geometry, algebraic topology, differential geometry, number theory, operator theory, group theory and measure theory are mature. We shall now supplement this criterion by saying that a mathematical theory is dynamic if it contains at least 50 important unsolved problems. It is possible that some of the mature mathematical theories are no longer dynamic, although this can probably only be decided by a researcher who works in this particular field. In many ways, unsolved problems are more interesting, exciting and stimulating than established results.

This article discusses some unsolved open problems for sequential effect algebras (SEAs). The theory of effect algebras and SEAs is too new to be mature. However, we believe that it is entering a dynamic phase. We certainly do not intend to introduce 50 open problems. However, we shall present 25 that we consider interesting and important. We also believe that effect algebra researchers can

[1] Department of Mathematics, University of Denver, Denver, Colorado; e-mail: sgudder@math.du.edu

provide many more. In order to understand these problems we shall first review some of the definitions and basic results for SEAs. We shall also discuss physical motivation and partial results.

2. BASIC DEFINITIONS AND RESULTS

Physically, an effect represents a two-valued measurement that may be fuzzy (or unsharp). An effect algebra is a collection of effects on which a parallel or statistical combination of certain elements is defined. To be precise, an *effect algebra* is a system $(E, 0, 1, \oplus)$ where 0 and 1 are distinct elements of E and $\oplus$ is a partial binary operation on E satisfying:

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(EA2) If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(EA3) For every $a \in E$, there exists a unique $a' \in E$ such that $a \oplus a' = 1$.
(EA4) If $a \oplus 1$ is defined, then $a = 0$.

If $a \oplus b$ is defined, we write $a \perp b$. We define $a \leq b$ if there is a $c \in E$ such that $a \oplus c = b$. It can be shown that $(E, \leq, ')$ is a bounded involution poset. That is, $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$ and $a \leq b$ implies that $b' \leq a'$. It can also be shown that $a \perp b$ if and only if $a \leq b'$. An element $a \in E$ is *sharp* if $a \wedge a' = 0$. For further motivation and results on effect algebras, we refer the reader to Bennett and Foulis (1977), Dvurečenskij and Pulmannová (2000), Foulis and Bennett (1994), Giuntini and Greuling (1989) and Gudder (1998).

In order to describe series combinations of effects, we introduce a sequential product (Gudder; Gudder and Greechie, 2002; Gudder and Greechie; Gudder and Nagy, 2001) on an effect algebra. For a binary operation $\circ$, if $a \circ b = b \circ a$ we write $a \mid b$. A *sequential effect* algebra (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ \colon E \times E \to E$ is a binary operation satisfying:

(SEA1) The map $b \mapsto a \circ b$ is additive for every $a \in E$ (that is, if $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$).
(SEA2) $1 \circ a = a$ for every $a \in E$.
(SEA3) If $a \circ b = 0$, then $a \mid b$.
(SEA4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for every $c \in E$.
(SEA5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$ whenever $a \perp b$.

We call an operation satisfying (SEA1)–(SEA5) a *sequential product* on E. If $a \mid b$ for every $a, b \in E$, then E is a *commutative* SEA. Notice that if $\circ$ is a commutative binary operation on an effect algebra E, to test whether $\circ$ is a sequential product we need only show (SEA1), (SEA2) and

(SEA4′) $a \circ (b \circ c) = (a \circ b) \circ c$ for every $a, b, c \in E$.

We now briefly describe the most important examples of SEAs.

Example 1. For a Boolean algebra $\mathcal{B}$, define $a \perp b$ if $a \wedge b = 0$ and in this case $a \oplus b = a \vee b$. Then $(\mathcal{B}, 0, 1, \oplus)$ is an effect algebra and all its elements are sharp. Under the unique sequential product $a \circ b = a \wedge b$, $\mathcal{B}$ becomes a SEA (Gudder and Greechie, 2002).

Example 2. For $[0, 1] \subseteq \mathbb{R}$ define $a \perp b$ if $a + b \leq 1$ and in this case $a \oplus b = a + b$. Then $([0, 1], 0, 1, \oplus)$ is an effect algebra whose only sharp elements are 0, 1. Under the unique sequential product $a \circ b = ab$, $[0, 1]$ becomes a SEA (Gudder and Greechie, to appear).

Example 3. Let $X \neq \emptyset$ and let $\mathcal{F} \subseteq [0, 1]^X$. We call $\mathcal{F}$ a *fuzzy set system* on X if the functions $0, 1 \in \mathcal{F}$, $1 - f \in \mathcal{F}$ whenever $f \in \mathcal{F}$, $f + g \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$ with $f + g \leq 1$ and $fg \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$. Then $(\mathcal{F}, 0, 1, \oplus)$ is an effect algebra with $f \oplus g = f + g$ for $f + g \leq 1$. The sharp elements of $\mathcal{F}$ are the characteristic functions. If $\mathcal{F} = [0, 1]^X$, we call $\mathcal{F}$ a *full* fuzzy set system. Under the sequential product $f \circ g = fg$, $\mathcal{F}$ becomes a SEA. If $\mathcal{F}$ is full, this sequential product is unique (Gudder and Greechie, to appear).

The previous examples were commutative SEAs while the next example is noncommutative.

Example 4. Let H be a Hilbert space and let $\mathcal{E}(H)$ be the set of self-adjoint operators on H satisfying $0 \leq A \leq I$. For $A, B \in \mathcal{E}(H)$, we define $A \perp B$ if $A + B \in \mathcal{E}(H)$ and in this case $A \oplus B = A + B$. Then $(\mathcal{E}(H), 0, I, \oplus)$ is an effect algebra. The sharp elements of $\mathcal{E}(H)$ consist of the set of projection operators $\mathcal{P}(H)$ on H. Under the sequential product $A \circ B = A^{1/2}BA^{1/2}$, $\mathcal{E}(H)$ becomes a SEA. Unlike the previous examples, it is not obvious that $\circ$ is indeed a sequential product. It is easy to show that (SEA1)–(SEA3) hold but verifying (SEA4) and (SEA5) are more difficult. For example, if $A \mid B$ it is not obvious that $A \mid B'$. This would say that $A^{1/2}BA^{1/2} = B^{1/2}AB^{1/2}$ implies that

$$A^{1/2}(I - B)A^{1/2} = (I - B)^{1/2}A(I - B)^{1/2} \tag{2.0}$$

The second part of (SEA4) would say that $A^{1/2}BA^{1/2} = B^{1/2}AB^{1/2}$ implies that

$$A^{1/2}B^{1/2}CB^{1/2}A^{1/2} = (A^{1/2}BA^{1/2})^{1/2}C(A^{1/2}BA^{1/2})^{1/2} \tag{2.1}$$

for every $C \in \mathcal{E}(H)$. However, applying Theorem 2.1 (Gudder and Nagy, 2001) which states that $A^{1/2}BA^{1/2} = B^{1/2}AB^{1/2}$ if and only if $AB = BA$, (SEA4) and (SEA5) easily follow.

3. OPEN PROBLEMS

We now state 25 open problems for SEAs. Some of these problems will be accompanied with comments and background material.

Problem 1. *We have seen in Example 3 that $f \circ g = fg$ is a sequential product on a fuzzy set system $\mathcal{F}$ and it is unique if $\mathcal{F}$ is full. Is this the only sequential product for arbitrary $\mathcal{F}$?*

Problem 2. *Is $A \circ B = A^{1/2}BA^{1/2}$ the only sequential product on $\mathcal{E}(H)$?*

A SEA E is a *σ-SEA* if for any increasing sequence $a_1 \leq a_2 \leq \cdots$ in E, the least upper bound $\vee a_i$ exists in E. We denote the set of sharp elements in a SEA E by E_S. A SEA E is *sharply dominating* if for every $a \in E$ there exists a least element $\widehat{a} \in E_S$ such that $a \leq \widehat{a}$ (Gudder, 1998).

Problem 3. *It can be shown that every σ-SEA is sharply dominating* (Gudder and Greechie, 2002). *Is every SEA sharply dominating?*

Problem 4. *If E is a SEA, then it can be shown that E_S is an orthomodular poset* (Gudder and Greechie, 2002). *Is every orthomodular poset isomorphic to E_S for some SEA E?*

Problem 5. *If E is a sharply dominating SEA, then it can be shown that E_S is an orthomodular lattice. Is every orthomodular lattice isomorphic to E_S for some sharply dominating SEA E?*

Problem 6. *In a SEA, does $a = b \circ a \oplus b' \circ a$ imply that $a \mid b$? (This result holds in $\mathcal{E}(H)$* (Gudder and Nagy, 2001).*)*

Problem 7. *In a SEA are the following statements equivalent? (i) $a \circ b = b$, (ii) $b \circ a = b$, (iii) $a \circ b = b \circ a = b$? (This result holds in $\mathcal{E}(H)$* (Gudder and Nagy, 2001).*)*

Problem 8. *For a SEA E does $a \circ (b \circ c) = (a \circ b) \circ c$ for every $c \in E$ imply that $a \mid b$? (This result holds in $\mathcal{E}(H)$* (Gudder and Nagy, 2001).*)*

Problem 9. *An effect a is* almost sharp *if $a = p \circ q$ for some sharp elements p and q. Characterize the almost sharp elements of a SEA. (This has been done for $\mathcal{E}(H)$.)*

Problem 10. *If a is almost sharp is $a^n = a \circ \cdots \circ a$ (n terms) almost sharp? (This result holds in $\mathcal{E}(H)$.)*

Problem 11. *Characterize effects of the form*

$$a = p_1 \circ \cdots (p_{n-2} \circ (p_{n-1} \circ p_n))$$

where the p_i, $i = 1, \ldots, n$, are sharp. (This has been done for $\mathcal{E}(H)$.)

Let E and F be SEAs. A *morphism from E to F* is an additive map $\phi \colon E \to F$ such that $\phi(1) = 1$ and $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all $a, b \in E$. A morphism $\phi \colon E \to [0, 1]$ is called a *multiplicative state*. We say that E and F are *isomorphic* if there exists a bijective morphism $\phi \colon E \to F$ such that ϕ^{-1} is a morphism. A solution to the next result would be important for the foundations of quantum mechanics.

Problem 12. *Characterize the SEAs that are isomorphic to $\mathcal{E}(H)$.*

A set of multiplicative states Ω on E is *order-determining* if $\omega(a) \leq \omega(b)$ for all $\omega \in \Omega$ implies that $a \leq b$.

Problem 13. *Give an algebraic characterization of SEAs that are isomorphic to a fuzzy set system. (A necessary but not sufficient condition is commutativity. Necessary and sufficient conditions are commutativity and an order determining set of multiplicative states* (Gudder and Greechie, 2002).*)*

Problem 14. *Give an algebraic characterization of SEAs that are isomorphic to a full fuzzy set system.*

For many algebraic structures, the concept of an ideal is very important. Ideals are frequently kernels of morphisms and are useful for constructing representations of the structures. The next problem, which is admittedly rather vague, asks whether such a concept may be applied for SEAs.

Problem 15. *Is there a useful definition of an ideal in a SEA?*

Let E, F, G be SEAs. A SEA *bimorphism* $\beta \colon E \times F \to G$ satisfies:

(i) $\beta(1, 1) = 1$,

(ii) $\beta(a, \cdot)$ is additive for every $a \in E$,

(iii) $\beta(\cdot, b)$ is additive for every $b \in F$,

(iv) $\beta(a \circ b, c \circ d) = \beta(a, c) \circ \beta(b, d)$ for every $a, b \in E, c, d \in F$.

A SEA *tensor product* of E and F is a pair (T, τ) where T is a SEA and $\tau: E \times F \to T$ is a SEA bimorphism such that

(1) Every element of T has the form $\tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$.
(2) If $\beta: E \times F \to G$ is a SEA bimorphism, there exists a morphism $\phi: T \to G$ such that $\beta = \phi \circ \tau$.

Theorem 1. *If E and F are commutative SEAs, then the SEA tensor product of E and F exists if and only if there exists a SEA bimorphism on $E \times F$ (*Gudder, to appear*).*

Problem 16. *Does Theorem 1 hold for an arbitrary SEA?*

Let A be a subset of a SEA E. We say that A is *commutative* if $a \mid b$ for all $a, b \in A$. We say that A is a *sub-SEA* of E if A is a sub-effect algebra of E and $a \circ b \in A$ whenever $a, b \in A$. The SEA $\overline{A}$ *generated* by A is the smallest sub-SEA of E that contains A. Of course, A is a sub-SEA of E if and only if $A = \overline{A}$. The *commutant* of A is defined by

$$C(A) = \{b \in E: b \mid a \text{ for all } a \in A\}$$

It is clear that $C(A)$ is a sub-SEA of E and that $A \subseteq C[C(A)]$.

Problem 17. *If A is commutative, is $\overline{A}$ commutative?*

Problem 18. *If A is commutative, is $C[C(A)]$ commutative?*

Problem 19. *Characterize the SEAs E such that $\overline{A} = C[C(A)]$ for all $A \subseteq E$.*

We say that b is a *square root* of a if $b^2 = a$.

Problem 20. *If a square root of a exists, is it unique?*

Problem 21. *Characterize the SEAs E such that a square root of a exists for all $a \in E$.*

Problem 22. *If $a \perp b$, is it the case that $a \circ b \perp a \circ b$?*

Problem 23. *If $a \perp b$ and $a \circ b \perp a \circ b$, is it the case that $2a \circ b \le a^2 \oplus b^2$?*

The next problem already appeared in Gudder and Nagy (2001) but it probably deserves repeating. Physically, the sequential product $a \circ b$ corresponds to a measurement in which a is performed first and b is performed second. In a noncommutative SEA, $\circ$ need not be associative so that $a \circ (b \circ c) \neq (a \circ b) \circ c$ in general. For example, in $\mathcal{E}(H)$, Eq. (2.1) does not necessarily hold. But this states that performing a first and then performing $b \circ c$ does not coincide with performing $a \circ b$ first and then performing c.

Problem 24. *Is there an actual experiment which verifies that $a \circ (b \circ c) \neq (a \circ b) \circ c$?*

Our last problem is related to Problem 6. A map $L: \mathcal{E}(H) \to \mathcal{E}(H)$ of the form $L(B) = \Sigma_{i=1}^{n} A_i \circ B$, where $A_i \in \mathcal{E}(H)$, $i = 1, \ldots, n$, satisfy $\Sigma A_i = I$ is called a *generalized Lüders* map (Gudder and Nagy, 2001). We say that B is an *L-fixed point* if $L(B) = B$. Fixed points are important in quantum measurement and quantum information theory. It can be shown that for $n = 2$, $L(B) = B$ if and only if $B \mid A_i, i = 1, 2$. Moreover, if $\dim H < \infty$ then for any n, $L(B) = B$ if and only if $B \mid A_i, i = 1, \ldots, n$. However, if $\dim H = \infty$ then this last result does not hold. The proof of this is nonconstructive and requires that $n \geq 5$. This suggests the following problem.

Problem 25. *Give a concrete example of a generalized Lüders map with $n = 3$ such that $L(B) = B$ does not imply that $B \mid A_i, i = 1, 2, 3$.*

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